1.6-1.7b Leslie's age structured model and Perron-Frobenius

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Assumptions:

- Population is closed to migration
- Only females are modeled directly
- Divide population into m fixed age groups, where m is the last reproductive age.
- As $t \to t+1$, individuals age from $i \to i+1$
- Individuals in the same age group have the same reproduction rate.

Mathematically

- $x_i(t)$ = number of females in ith age group at time t.
- b_i = average number of newborn females produced by one female in the ith age group that survives their birth time interval.
- s_i = fraction of ith age group that survives until $(i+1)$th age.

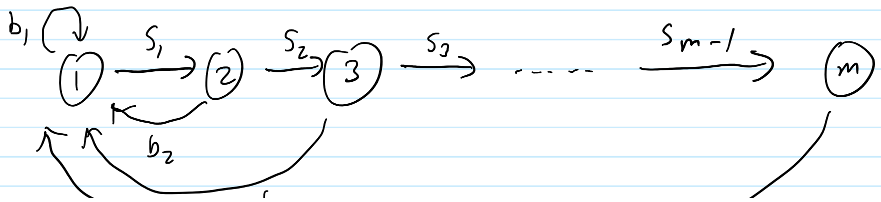
Then $x_1(t+1) = \sum_{i=1}^{m} b_i x_i(t)$ and $x_{i+1}(t+1) = s_i x_i(t)$, $i > 1$

Equivalently,

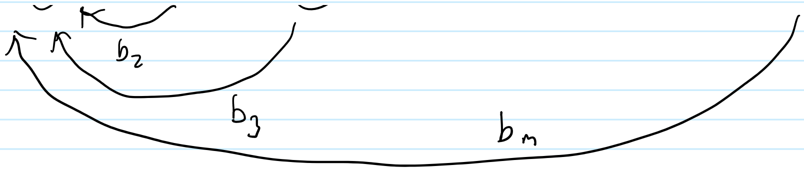
$$X(t+1) = \begin{pmatrix} x_1(t+1) \\ \vdots \\ \vdots \\ x_m(t+1) \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 & \cdots & \cdots & b_m \\ \hline s_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & s_{m-1} & 0 \end{pmatrix}}_{L = \text{Leslie matrix (aka projection matrix)}} \begin{pmatrix} x_1(t) \\ \vdots \\ \vdots \\ x_m(t) \end{pmatrix} = L X(t)$$

In general $X(t) = L^t X(0)$.

Life cycle graph



(directed graph or digraph)



| or digraph /

Def. A **permutation matrix** $P \in \mathbb{R}^{m \times m}$ has exactly one 1 in every row and col, and 0's everywhere else. **Note:** $P^T P = I$.

Ex.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \cancel{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \quad \cancel{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Def. 1.10 Let $A \in \mathbb{R}^{m \times m}$. A is **reducible** if $\exists$ permutation matrix P s.t.

$$P^T A P = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right],$$

where A_{11}, A_{22} are square matrices of nonzero size.

Otherwise, A is **irreducible**.

Def. Let $A \in \mathbb{R}^{m \times m}$ be a nonnegative matrix $(A \geq 0)$. Then **digraph(A)** is the directed graph with m nodes labelled $\{1, \dots, m\}$ and edge $i \to j$ with weight a_{ji} if $a_{ji} \neq 0$.

Def. 1.11 If digraph(A) has a directed path from i to j $\forall\, i, j \in \{1, \dots, m\}$, then digraph(A) is **strongly connected**.

Thm 1.2 digraph(A) is strongly connected iff A is irreducible.

Case 1: Suppose A is reducible.

WLOG, $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ If nonzero entry a_{ji}, that means there is an edge $i \to j$.

Consider A^k If nonzero entry $(A^k)_{ij}$, that means there is

Consider A^k. If nonzero entry $(A^k)_{ij}$, that means there is a path of length k from i to j.

In order for digraph(A) to be strongly connected, we need a path of some length k from i to j for all i, j.

But $A^2 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^2 & A_{11}A_{12} + A_{12}A_{22} \\ \underline{0} & A_{22}^2 \end{pmatrix}$

and by induction $A^k = \begin{pmatrix} A_{11}^k & * \\ 0 & A_{22}^k \end{pmatrix}$, so $\forall k$, there's no path from node 1 to node m.

$\uparrow$ $(A^k)_{m1}$

Thus, if A is reducible, then it is not strongly connected.

Case 2: Suppose digraph(A) is not strongly connected.

Pick an arbitrary node and label it 1. WLOG, assume that there exists no path from 1 to some other node. Can partition the nodes by whether or not, you can reach it from node 1.

But then, clearly can't reach it in 1 step.

So

$$A = \left(\begin{array}{c|c} \begin{matrix} * & \cdots & * \\ * & & \vdots \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \\ \hline \begin{matrix} 0 & & \\ \vdots & 0 & \\ 0 & & \end{matrix} & * \end{array}\right)$$

is block diagonal under that node relabelling where the first half are nodes reachable from node 1.

$\Rightarrow$ A must be reducible. $\boxed{\diagup}$

Thm 1.3 (Frobenius) If $A \in \mathbb{R}^{m \times m}$ is irreducible and nonnegative $(A \geq 0)$, then $\lambda = \rho(A)$ is a dominant eigenvalue of multiplicity 1. And, λ has a corresponding eigenvector with positive components.

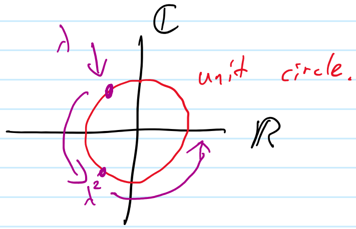
Thm 1.4 (Perron) If $A \in \mathbb{R}^{m \times n}$ is positive, then $\lambda = \rho(A)$ is a strictly dom. eigenvalue of multiplicity 1. And, λ has

Thm 1.4 (Perron) If $A \in \mathbb{R}^{m\times m}$ is positive, then $\lambda = \rho(A)$ is a strictly dom. eigenvalue of multiplicity 1. And, λ has a corresponding eigenvector with pos. components.

proof. sketch: (for existence of strictly dom pos eigenvalue of positive A)

WLOG, assume $\rho(A)=1$ (if not, consider $A/\rho(A)$).

$\Rightarrow \exists$ eigenvalue λ with $|\lambda|=1$.



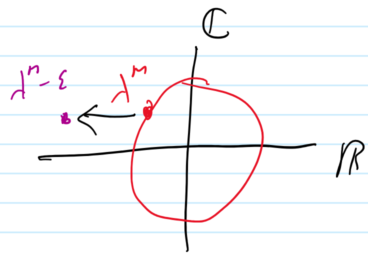
Suppose $\lambda \neq 1$. Then $\exists\, m \in \mathbb{Z}^+$ s.t. $A^m > 0$ and $\mathrm{Re}\,\lambda^m < 0$.

Let ε be half the smallest diagonal entry of A^m. (Note $\varepsilon > 0$ because $A^m > 0$)

Then define $T = A^m - \varepsilon I > 0$.

If $Ax = \lambda x$, then $A^m x = \lambda^m x$.

$\Rightarrow \lambda^m - \varepsilon$ is an eigenvalue of T.



But $|\lambda^m - \varepsilon| > 1$

Gelfand's formula

$\Rightarrow \rho(T) > 1$. However, $\rho(T) \le \rho(A^m) \le \rho(A)^m = 1$. A contradiction.

$\Rightarrow \lambda = 1$ and there are no other eigenvalues on the unit circle. $\boxtimes$

Def. 1.12 If $A \in \mathbb{R}^{m\times m}$ is irreducible and $A \geq 0$, and has h eigenvalues of maximum modulus, then A is **primitive** if $h=1$ and **imprimitive** if $h \neq 1$.

h is the **index of imprimitivity**.

Thm 1.5 If $A \in \mathbb{R}^{m\times m}$ and $A \geq 0$, then A is primitive iff $A^p > 0$ for some $p \in \mathbb{Z}^+$.

Def. The **inherent net reproductive number R_0** is the expected number of offspring for an individual over its lifetime.

Ex. $R_0 = b_1 + b_2 s_1 + b_3 s_1 s_2 + \cdots + b_m s_1 \cdots s_{m-1}$ in the Leslie model.

Thm 1.6 Consider a Leslie matrix L. If L is irreducible and primitive, then $\lambda_1 = \rho(L) > 0$. Furthermore,

$$\lambda_1 = 1 \quad \text{iff} \quad R_0 = 1,$$
$$\lambda_1 > 1 \quad \text{iff} \quad R_0 > 1,$$
$$\lambda_1 < 1 \quad \text{iff} \quad R_0 < 1.$$

Additionally, we have a stable age distribution V_1,

$$V_1 = \begin{pmatrix} 1 \\ \frac{s_1}{\lambda_1} \\ \vdots \\ \frac{s_1 s_2 \cdots s_{m-1}}{\lambda_1^{m-1}} \end{pmatrix},$$

where V_1 is the eigenvector associated with λ_1.

Thm 1.7 An irreducible Leslie matrix L is primitive iff the birth rates satisfy the following relationship,

$$\underbrace{\text{g.c.d.}}_{\text{greatest common divisor}} \{ i \mid b_i > 0 \} = 1.$$

Ex. 1.20

$$L = \begin{pmatrix} 0 & b_2 \\ s_1 & 0 \end{pmatrix}, \quad b_2, s_1 > 0.$$

$$\begin{tikzcd} 1 \arrow[r, "s_1"] & 2 \arrow[l, bend left, "b_2"] \end{tikzcd}$$

L satisfies the Frobenius theorem because the life cycle graph is strongly connected.

However, L is imprimitive.

↳ One way to see this is to use Thm. 1.5.

$$L = \begin{pmatrix} 0 & b_2 \\ s_1 & 0 \end{pmatrix} \qquad L^2 = \begin{pmatrix} 0 & b_2 \\ s_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & b_2 \\ s_1 & 0 \end{pmatrix} = \begin{pmatrix} b_2 s_1 & 0 \\ 0 & b_2 s_1 \end{pmatrix}$$

Turns out that L^p will always have zeros.

↳ Another way to see it is to explicitly compute eigenvalues

$$\lambda_{1,2} = \pm \sqrt{b_2 s_1}$$

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↳ Yet another way to see this is using Thm 1.7.

$$g.c.d.\{2\} = 2.$$

$\Rightarrow L$ is imprimitive.